Padovan heaps

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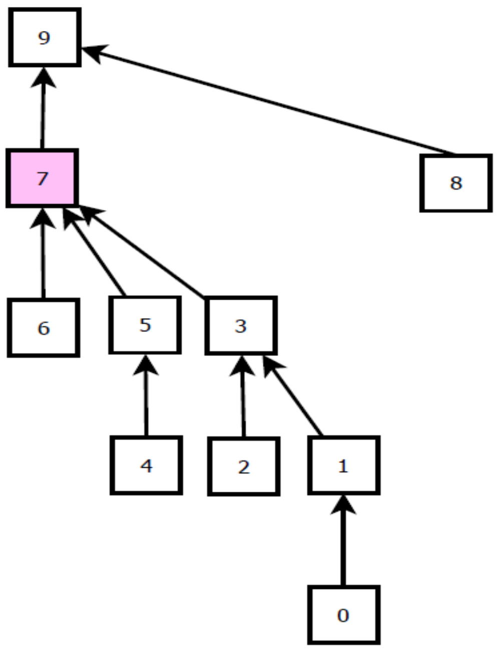
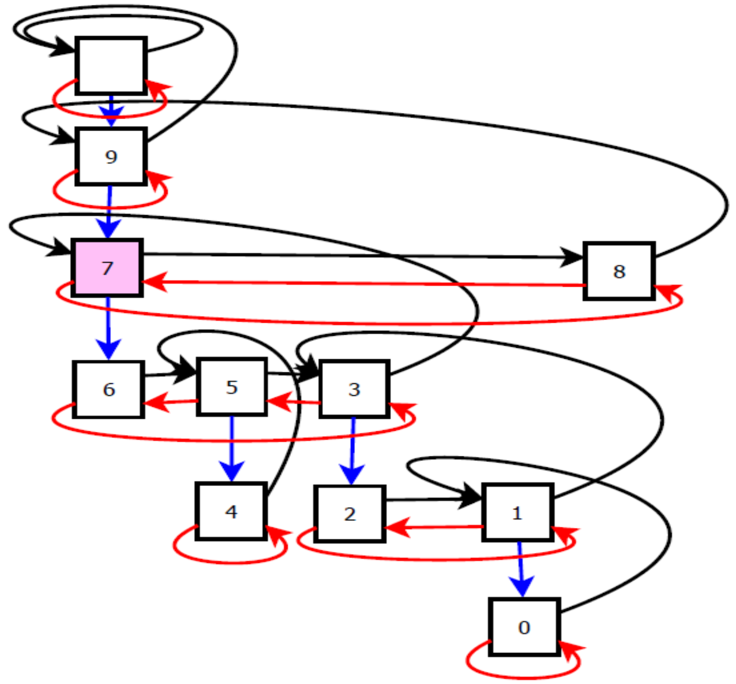




Main principle – superexpensive comparisons

- We compare as late as possible (FindMin does almost all work).
- We never forget results of comparisons.
- Organization of comparisons should not take asymptotically more time than comparisons.

Invariant of c-q narrow trees

There will be rank defined for a heap element v denoted $rank_v$. (New element gets rank 0).

- For c>0 and 1<q$\leq$2 holds:

Subtree of ancestors of an element of rank k has size at least cq^k.

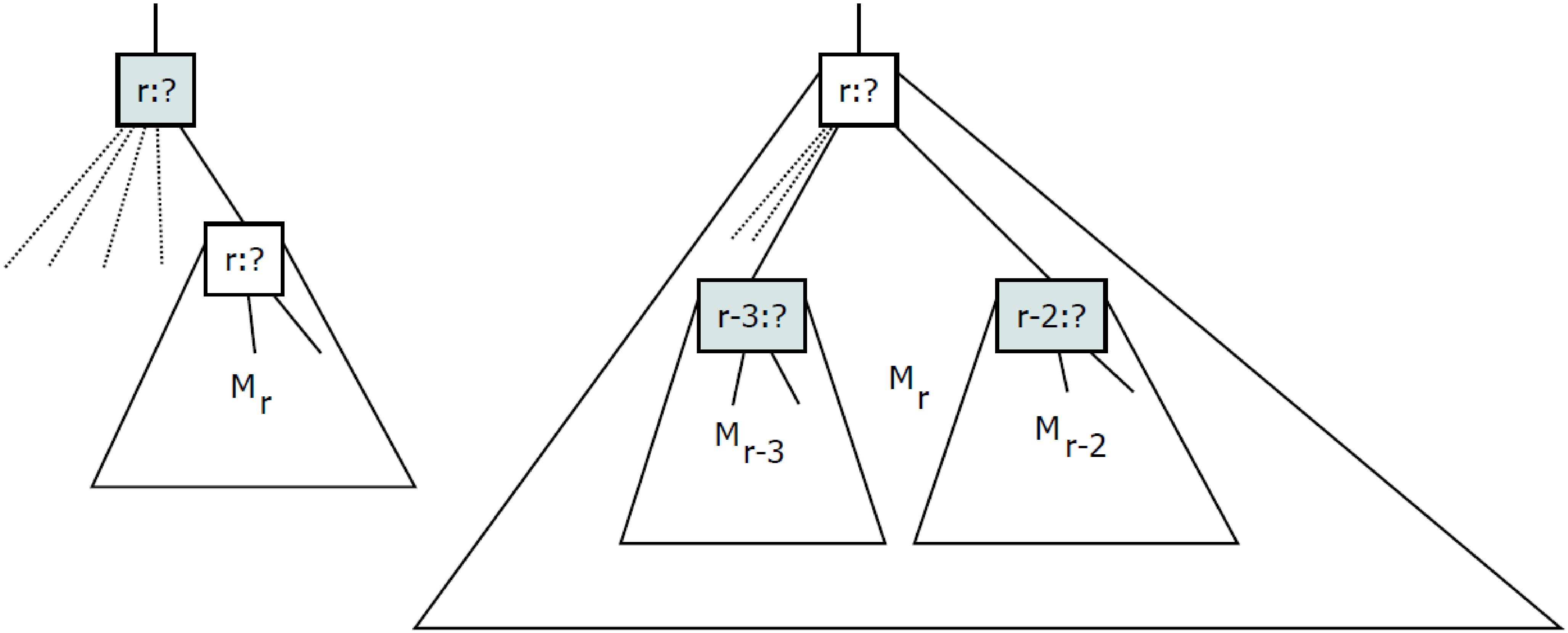
Saving one pointer per element

- Decrement in Fibonacci heaps require pointer to successor (parent) to allow rank decrement propagation.
- We save the space by replacing nil pointers at the right ends of lists by pointer to successor (parent). We would not maintain pointers to successors (parents) on other places.
- In that case we cannot propagate rank decrements except at the right end of the list where we can access the successor.
- We would propagate rank recomputation only at the rightmost two elements (elements of highest rank among white and black).
- Worst case time for DeleteMin will be O(1) again.

$rank\_v$ and $wrank\_v$

- Let us define $wrank = 1 + rank$ for a black element, and $wrank = \ rank$ for a white one. Let $wrank = -1$ for $null$ pointers (missing vertices).
- Predecessor (children) lists have all red on left, than $wrank$ increasing (when defined).
- Take among black and white predecessors last two ... w_0, w_1 from right (they could be missing).
  - S. Safe: $wrank_{w_1} + 1 = wrank_{w_0}$: $rank_v = wrank_{w_0} + 1$.
  - D. Dangerous: $wrank_{w_1} + 1 < wrank_{w_0}$ and w_0 is white: $rank_v = wrank_{w_0}$. (v cannot be white)
  - F. Forbidden: $wrank_{w_1} + 1 < wrank_{w_0}$ and w_0 is black: color w_0 yellow and recompute.

Minimal size recurrence



Narrowness

- Let M_k be minimal size of an ancestors tree of an element of rank k
- Following recurrence holds:

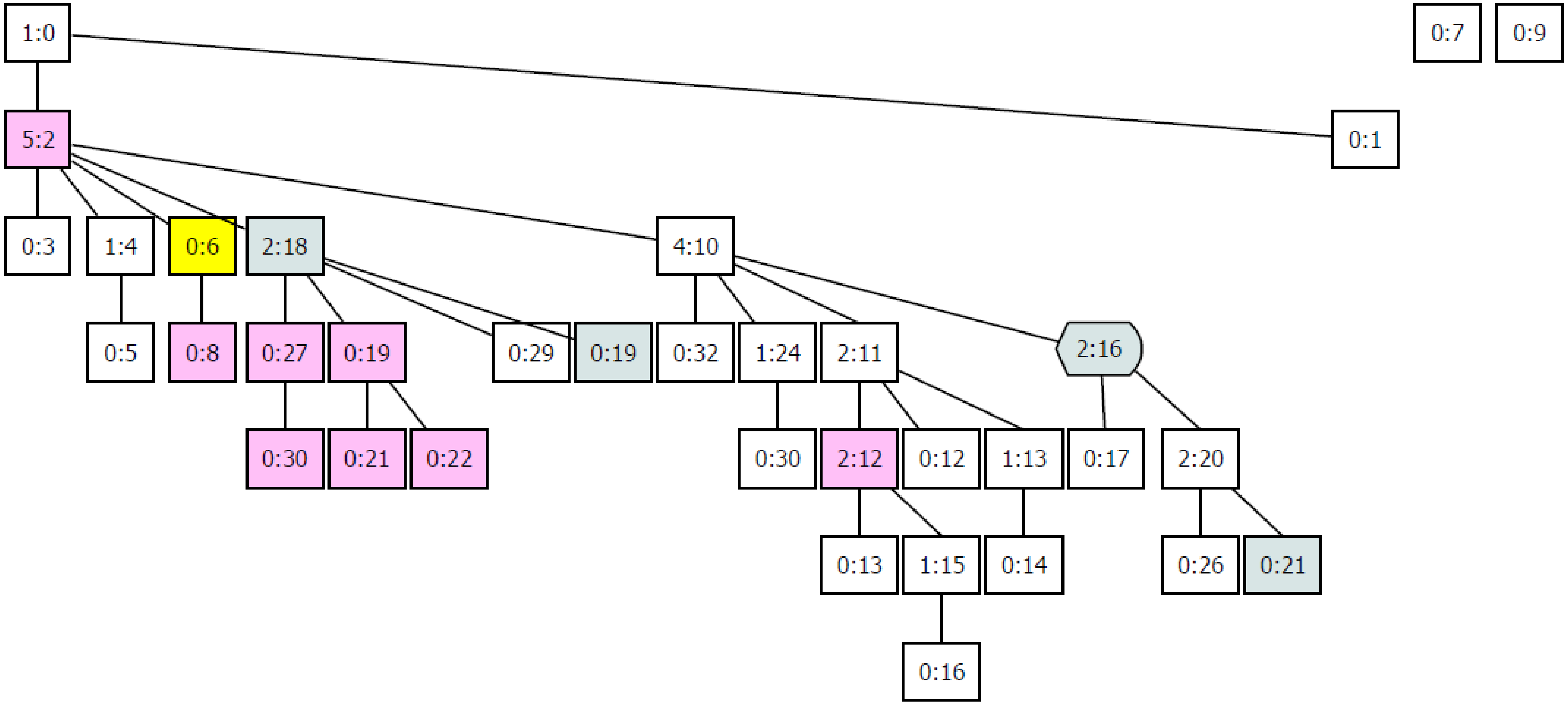
$$M_k = 1 + M_{k-2} + M_{k-3}$$

- This is closely related to Padovan sequence and c-q narrowness holds for q= $\sqrt[3]{\left(\frac{1}{2}\left(1+\sqrt{\frac{23}{27}}\right)\right)} + \sqrt[3]{\left(\frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)\right)} \approx 1.324718$
- $(q^3 = q + 1)$

Decrement fully colored

- We cut an element and if it was one of last two elements of a list we call cascading rank consolidation on it’s successor (parent).
- During cascade consolidation we start by computation of current rank. If there is no change, we are done.
- If rank of a white element drops by 1, it is colored black.
- If rank of a black element drops, it is colored yellow as well as if rank of a white element drops by at least 2.
- If an element is among last two of the list we continue the rank consolidation at the successor (parent).
- During rank calculation of an element we move all yellow predecessors(children) among right two elements to the left end. We color them red during it. If there is a red element among rightmost two, we know there is no more white and black element to the left in the list.

Picture with all colors (negated keys)



Summary

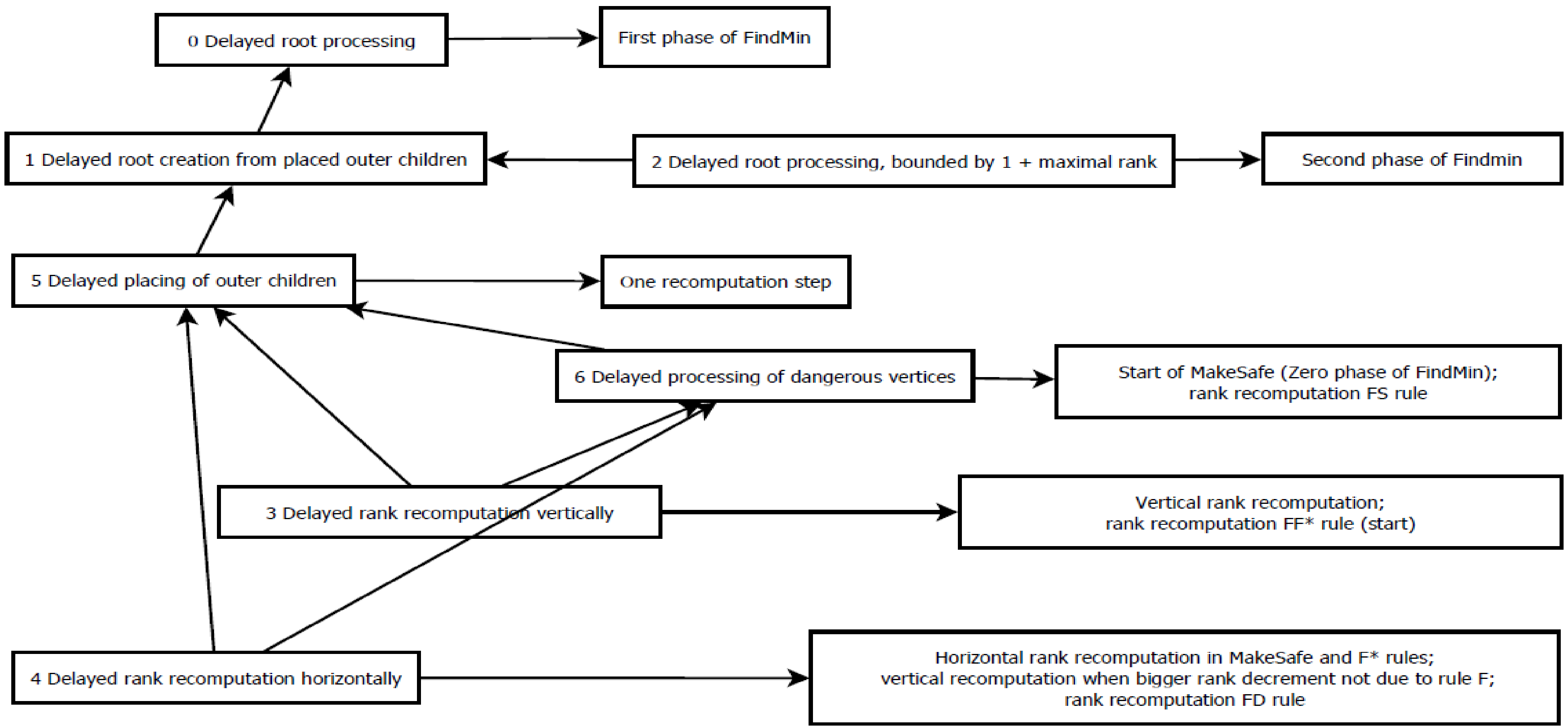
- Potential used in analysis is

$$\Phi_0 t_0 + \Phi_1 t_1 + \Phi_2 t_2 + 2\Phi_3 t_3 + \Phi_4 t_4 + \Phi_5 t_5 + \Phi_6 t_6,$$

where $t_0 \leq t_1 < t_2, t_1 < t_5 < t_6, t_5 + t_6 < t_3$, and $t_5 + t_6 < t_4$ are appropriate constants

- Φ_0 is number of candidates for minimum.
- Φ_2 is number of candidates for minimum, bounded by highest achievable rank+1.
- Φ_1, Φ_3, resp. Φ_5 is number of all red, black, resp. yellow descendants (children).
- Φ_4 is sum of differences between rank and number of white and black descendants (children) of an element.
- Φ_6 is number of dangerous vertices

The payment schema



Data structure's competition

- Supporter of one structure generates sequence of method calls, both structures invoke the methods and ratio of total times is the gain. The game is repeated with roles changed.
- Heaps according superexpensive comparison principle will won against standard implementation by $\theta(\log n)$ using prefix of a sequence

  i=0;Repeat (Insert(-i), Insert(-(i+1)), FindMin, DeleteMin, i++),

  because maximal rank achieved would be 4.
- Standard heaps could gain at most $\theta(1)$ in revenge.