H.P. WILLIAMS

LONDON SCHOOL OF
ECONOMICS

MODELS FOR SOLVING
THE
TRAVELLING SALESMAN PROBLEM

h.p.williams@lse.ac.uk

STANDARD FORMULATION OF THE (ASYMMETRIC) TRAVELLING SALESMAN PROBLEM

Conventional Formulation:

(cities *1,2, …, n*) (Dantzig, Fulkerson, Johnson) (1954). x_{ij} is a link in tour

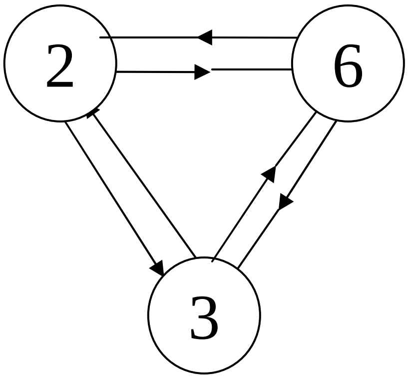
Minimise: $$\sum_{i,j} c_{ij} x_{ij}$$

subject to: $$\sum_{i} x_{ij} = 1 \quad \text{all } j$$

$$\sum_{j} x_{ij} = 1 \quad \text{all } i$$

$$\sum_{i,j \in S} x_{ij} \leq |S| - 1 \text{ all } S \subset \{2 \ldots, n\}$$

e.g.



$$x_{32} + x_{26} + x_{63}$$

$$+ x_{23} + x_{62} + x_{36} \leq 2$$

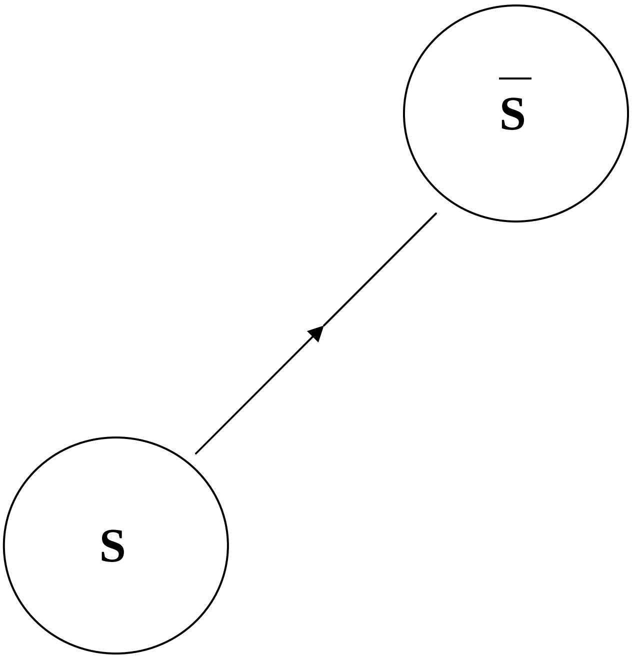
0(2^n) *Constraints* **= (2^{n-1} + n – 2)**

0(n^2) *Variables* **= n(n – 1)**

EQUIVALENT FORMULATION

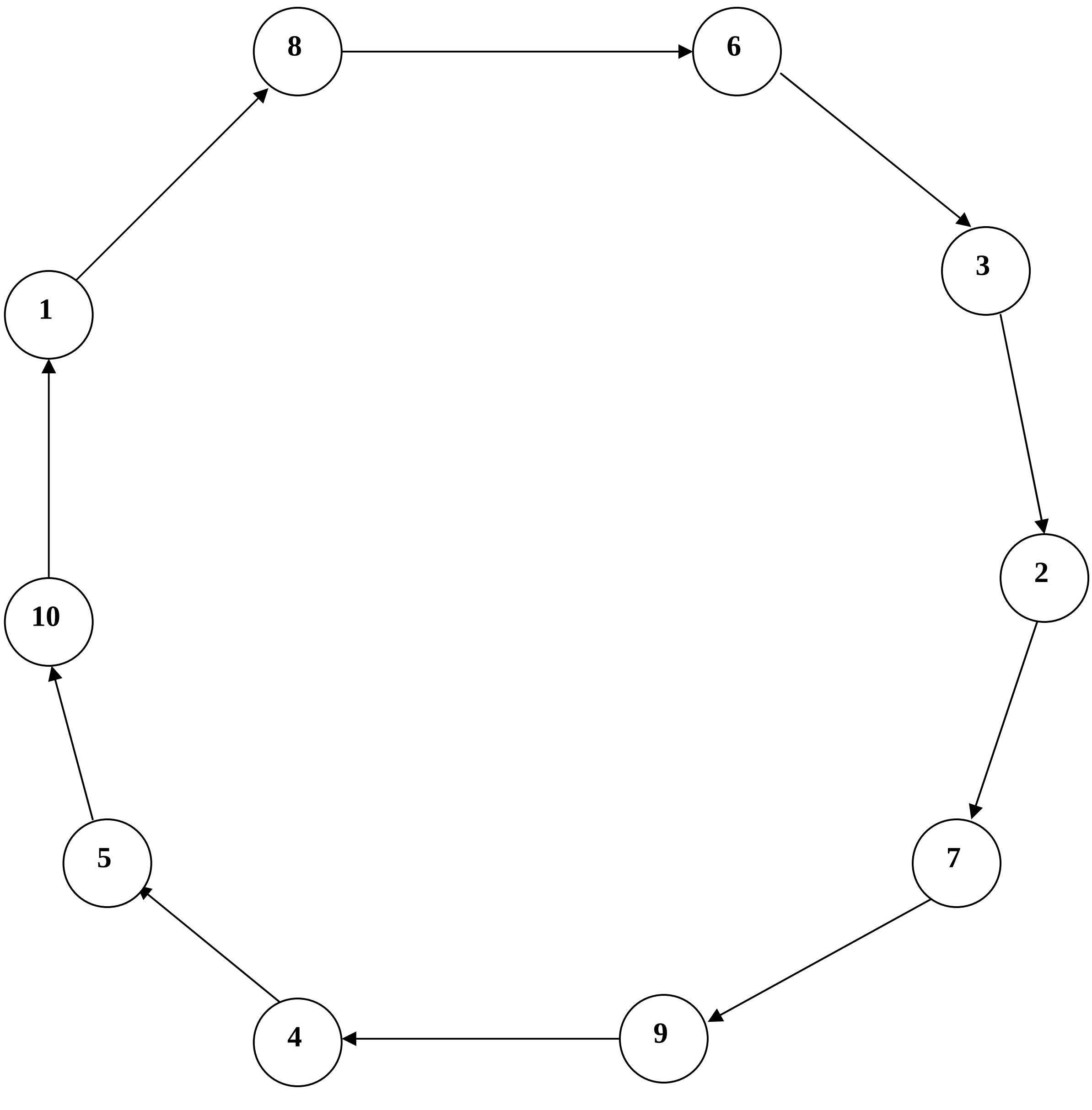
Replace subtour elimination constraints with

$$\sum_{\substack{i \in S \\ j \in \overline{S}}} x_{ij} \geq 1 \quad \textbf{all} \quad S \subset \{1, 2, ,..., n\}$$



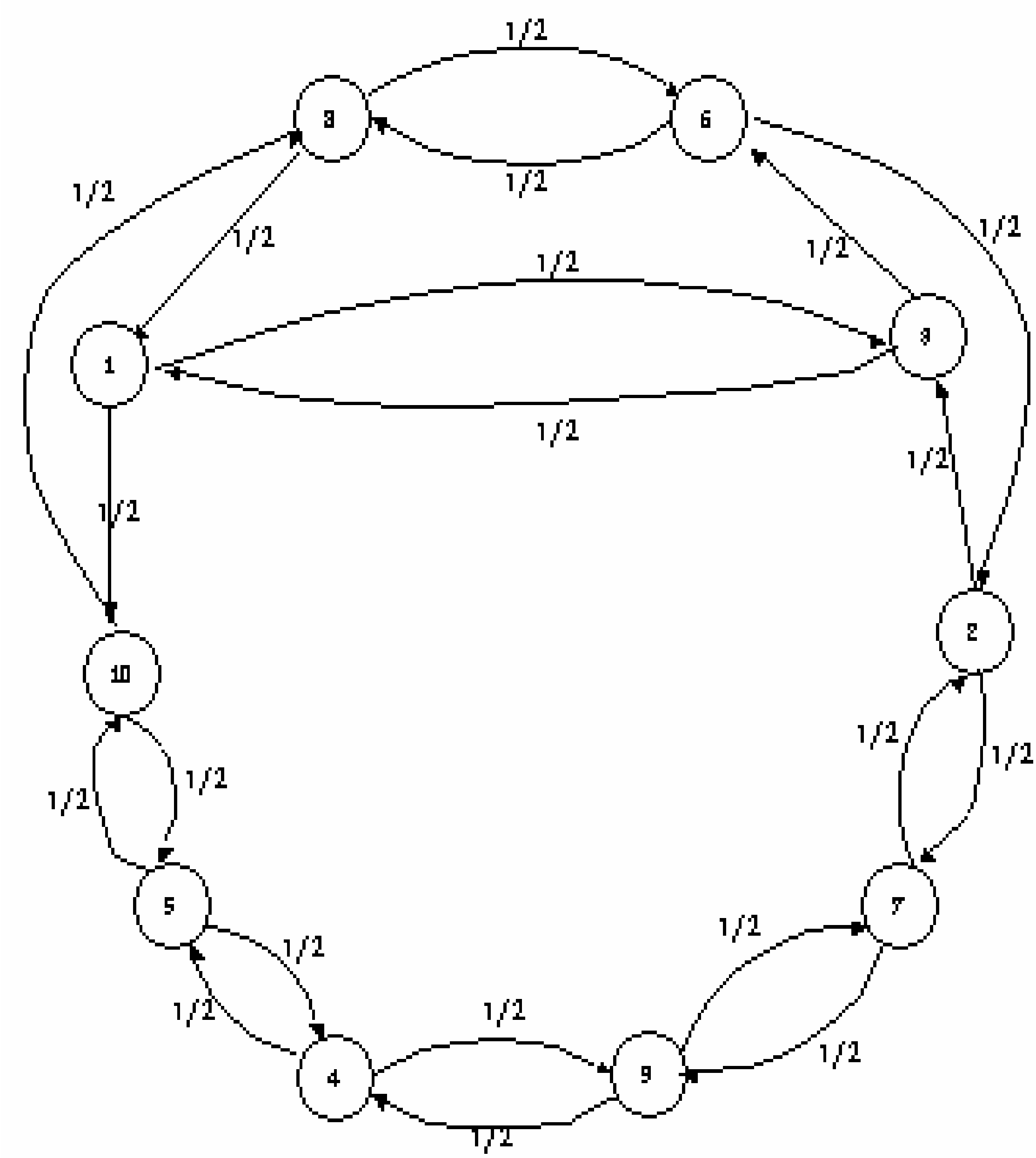
Add second set of constraints for all *i* in S and subtract from subtour elimination constraints for S

OPTIMAL SOLUTON TO A 10 CITY TRAVELLING SALESMAN PROBLEM



Cost = 881

FRACTIONAL SOLUTION FROM CONVENTIONAL (EXPONENTIAL) FORMULATION)



Cost = 878 (Optimal Cost = 881)

Sequential Formulation (Miller, Tucker, Zemlin (1960))

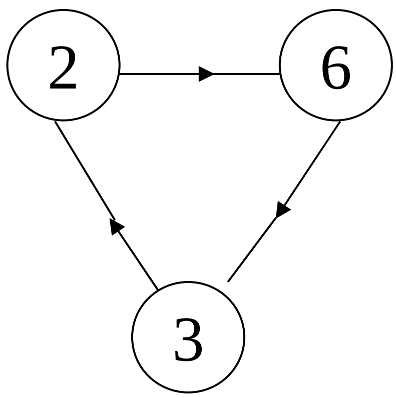
u_i **=** Sequence Number in which city i visited

Defined for $i = 2,3, \ldots, n$

Subtour elimination constraints replaced by

S: $u_i - u_j + nx_{ij} \leq n - 1 \quad i,j = 2,3, \ldots, n$

Avoids subtours
but allows total tours (containing city 1)



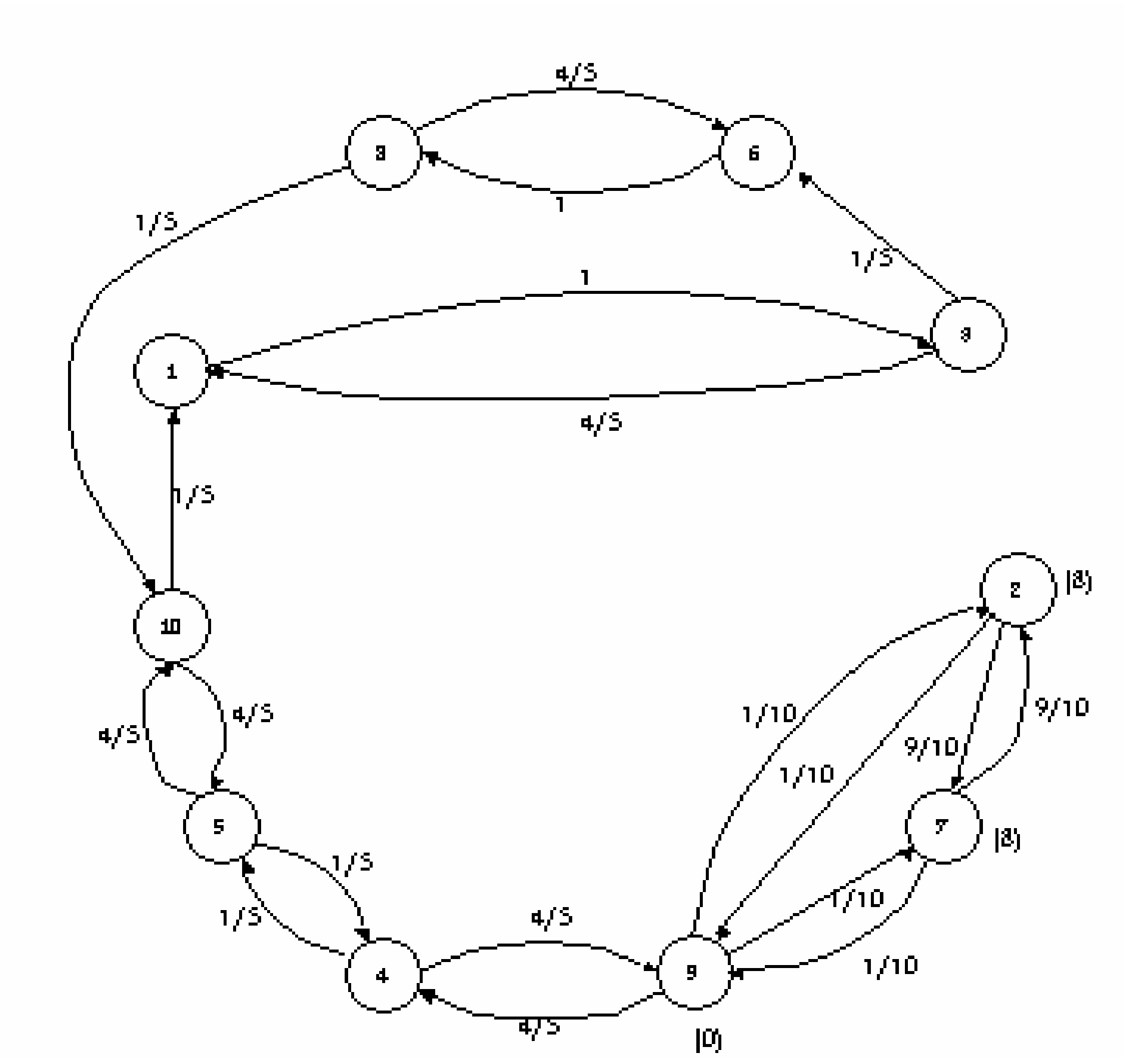
$$u_2 - u_6 + nx_{26} \leq n\text{-}1$$
$$u_6 - u_3 + nx_{63} \leq n\text{-}1$$
$$u_3 - u_2 + nx_{32} \leq n\text{-}1$$
$$\downarrow$$
$$3n \leq 3(n - 1)$$

$0(n^2)$	*Constraints*	=	$(n^2 - n + 2)$
$0(n^2)$	*Variables*	=	$(n - 1)(n + 1)$

Weak but can add 'Logic Cuts'

e.g. $u_k \geq 1 + x_{ij} + x_{jk} - x_{1j}$

FRACTIONAL SOLUTION FROM SEQUENTIAL FORMULATION



Subtour Constraints Violated : e.g. $\boldsymbol{x}_{27} + \boldsymbol{x}_{72} \not\leq 1$

Logic Cuts Violated: e.g. $u_9 \not\geq 1 + x_{27} + x_{79} - x_{17}$

Cost = $773\ ^3/_5$ (Optimal Cost = 881)

Flow Formulations

Single Commodity (Gavish & Graves (1978))

Introduce extra variables ('Flow' in an arc)

Replace subtour elimination constraints by

F1:

$$y_{ij} \le (n-1)x_{ij} \quad \text{all } i, j$$

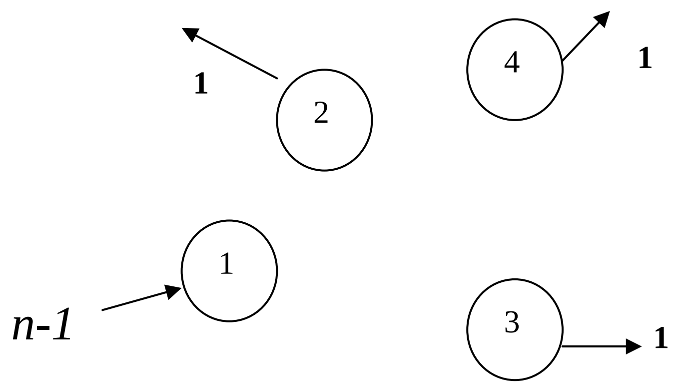
$$\sum_j y_{1j} = n-1$$

$$\sum_i y_{ij} - \sum_k y_{jk} = 1 \text{ all } j \ne 1$$

Can improve (F1') by amended constraints:

$$\boldsymbol{y}_{ij} \le (\boldsymbol{n}-2)\boldsymbol{x}_{ij} \qquad \text{all } i, j \ne 1$$

Network Flow formulation in y_{ij} variables over complete graph

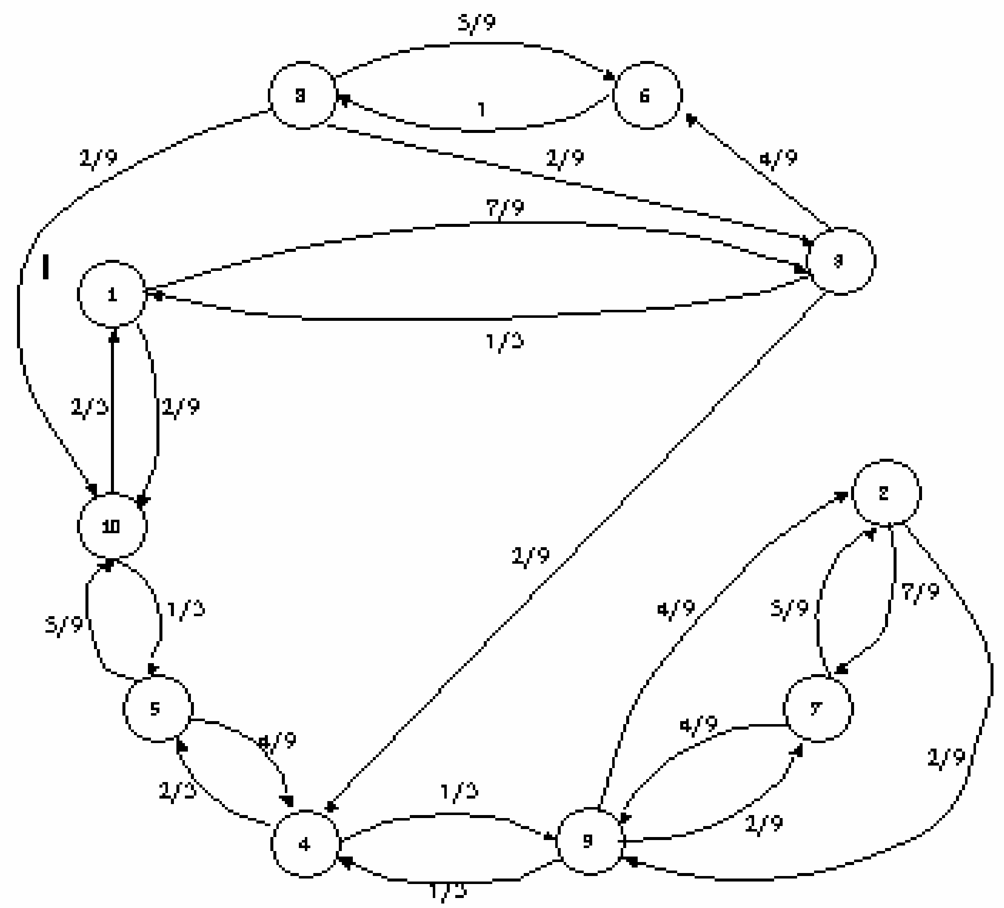


Graph must be connected. Hence no subtours possible.

$0(n^2)$ Constraints $= n(n+2)$

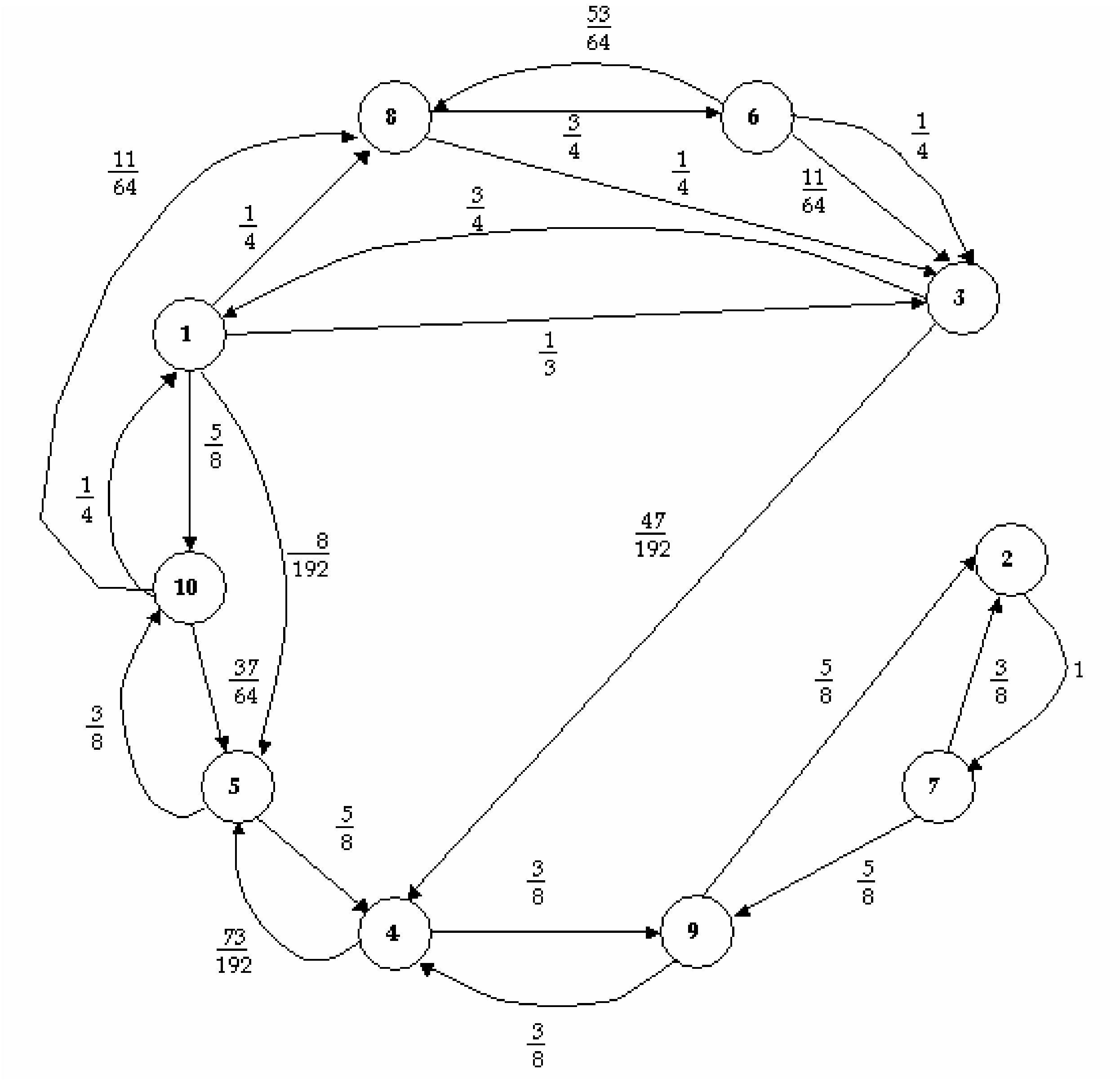
$0(n^2)$ Variables $= 2n(n-1)$

FRACTIONAL SOLUTION FROM SINGLE COMMODITY FLOW FORMULATION



Cost = $794\frac{2}{9}$ (Optimal solution = 881)

FRACTIONAL SOLUTION FROM MODIFIED SINGLE COMMODITY FLOW FORMULATION



Cost = $794\frac{43}{48}$ (Optimal solution = 881) (192=3x64)

Two Commodity Flow (Finke, Claus Gunn (1983))

y_{ij} is flow of commodity 1 in arc $i \rightarrow j$

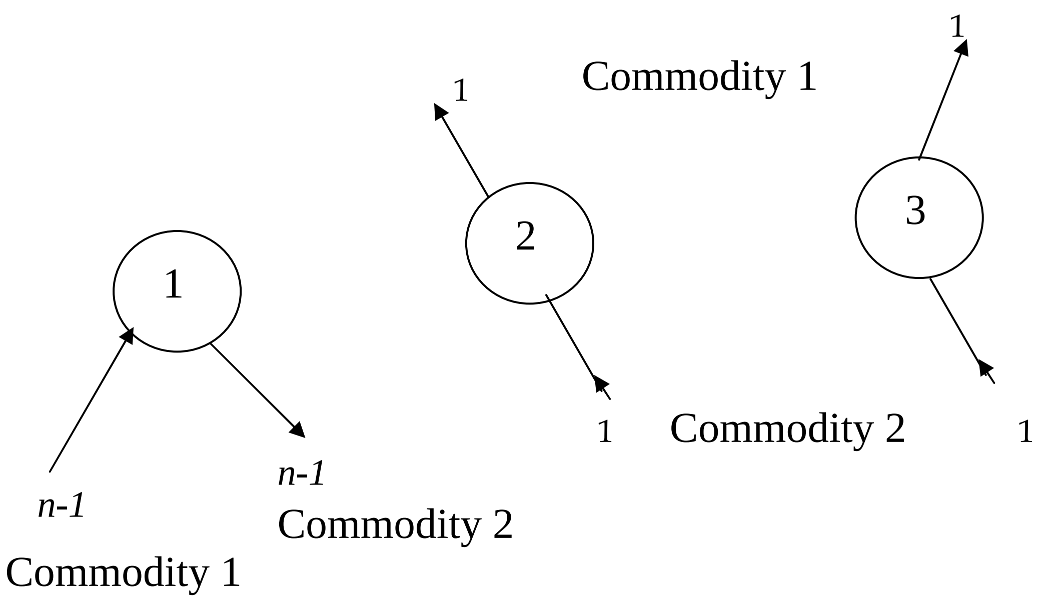
z_{ij} is flow of commodity 2 in arc $i \rightarrow j$

$$\sum_j y_{ij} - \sum_j y_{ji} \quad \begin{array}{ll} = -1 & i \neq 1 \\ = n-1 & i = 1 \end{array}$$

F2: $$\sum_j z_{ij} - \sum_j z_{ji} \quad \begin{array}{ll} = 1 & i \neq 1 \\ = -(n-1) & i = 1 \end{array}$$

$$\sum_j z_{ij} - \sum_j z_{ji} = n-1 \quad \text{all } i$$

$$y_{ij} + z_{ij} = (n-1)x_{ij} \quad \text{all } i, j$$



$0(n^2)$ Constraints $= n(n+4)$

$0(n^2)$ Variables $= 3n(n-1)$

Multi-Commodity (Wong (1980) Claus (1984))

"Dissaggregate" variables

y_{ij}^{k} is flow in arc destined for k

$y_{ij}^{k} \le x_{ij}$ all i, j, k

F3 $\sum_i y_{ik}^{k} = 1 \quad \sum_i y_{1i}^{k} = 1 \quad \sum_i y_{i1}^{k} = 0 \quad \sum_j y_{kj}^{k} = 0$ all k

$\sum_i y_{ij}^{k} = \sum_i y_{ji}^{k}$ all $j, k, j \ne 1, j \ne k$.

$0(n^3)$ Constraints $= n^3 - 2n^2 + 6n - 3$

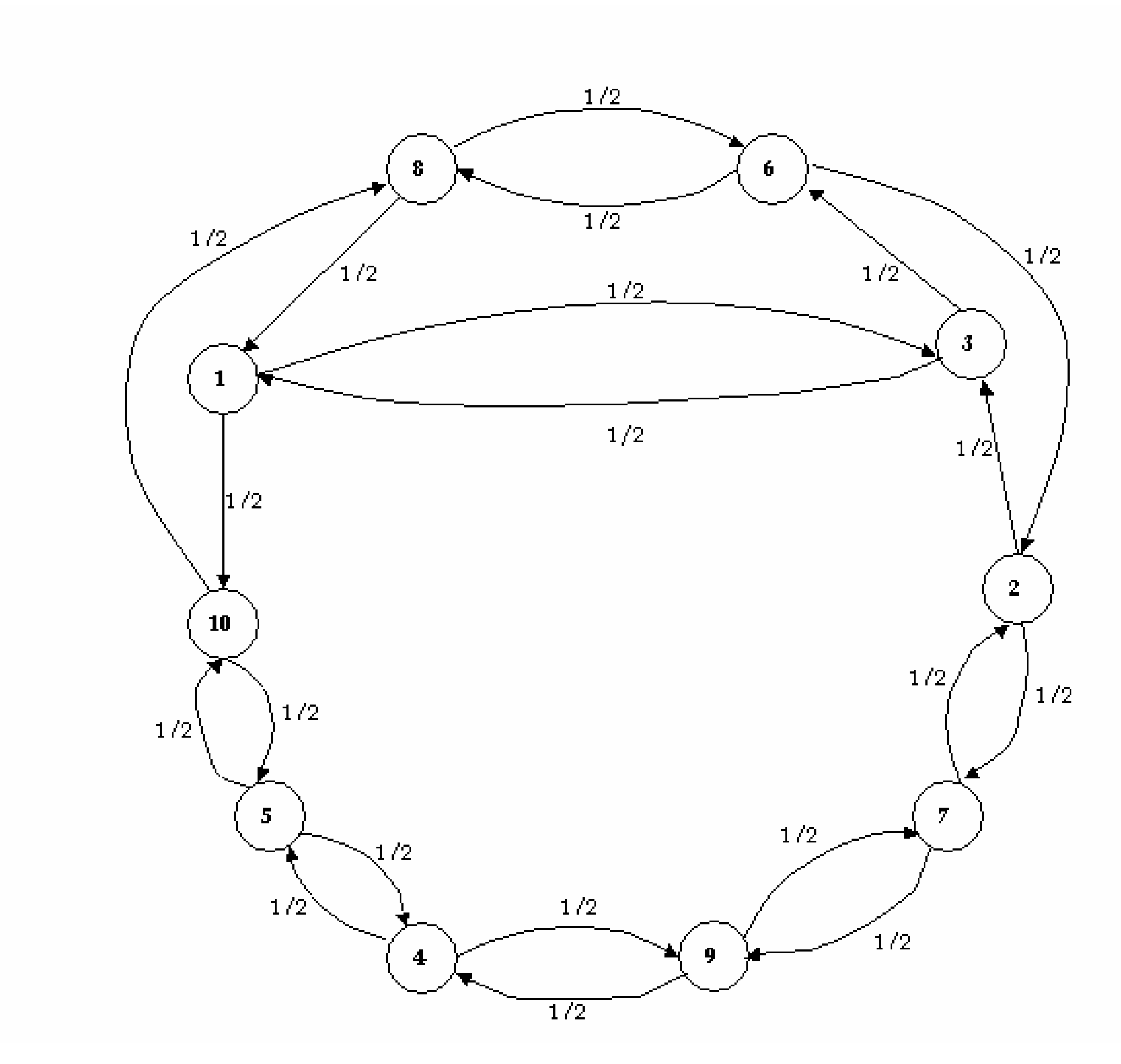
$0(n^3)$ Variables $= n^2(n-1)$

LP Relaxation of equal strength to Conventional Formulation.

But of polynomial size.

Tight Formulation of Min Cost Spanning Tree
+ (Tight) Assignment Problem

FRACTIONAL SOLUTION FROM MULTI COMMODITY FLOW FORMULATION (= FRACTIONAL SOLUTION FROM CONVENTIONAL (EXPONENTIAL) FORMULATION)



Cost = 878 (Optimal Cost = 881)

Stage Dependent Formulations

First (Fox, Gavish, Graves (1980))

= 1 if arc $\boldsymbol{i} \rightarrow \boldsymbol{j}$ traversed at stage *t*

= 0 otherwise

T1:

$$\sum_{i,j,t} y_{ij}^{t} = n$$

$$\sum_{j=1}^{n} \sum_{t=2}^{n} t y_{ij}^{t} - \sum_{j=1}^{n} \sum_{t=1}^{n} t y_{ji}^{t} = 1 \quad i = 2, 3 \ldots \mathrm{n}$$

(Stage at which ***i*** left 1 more than stage at which entered)

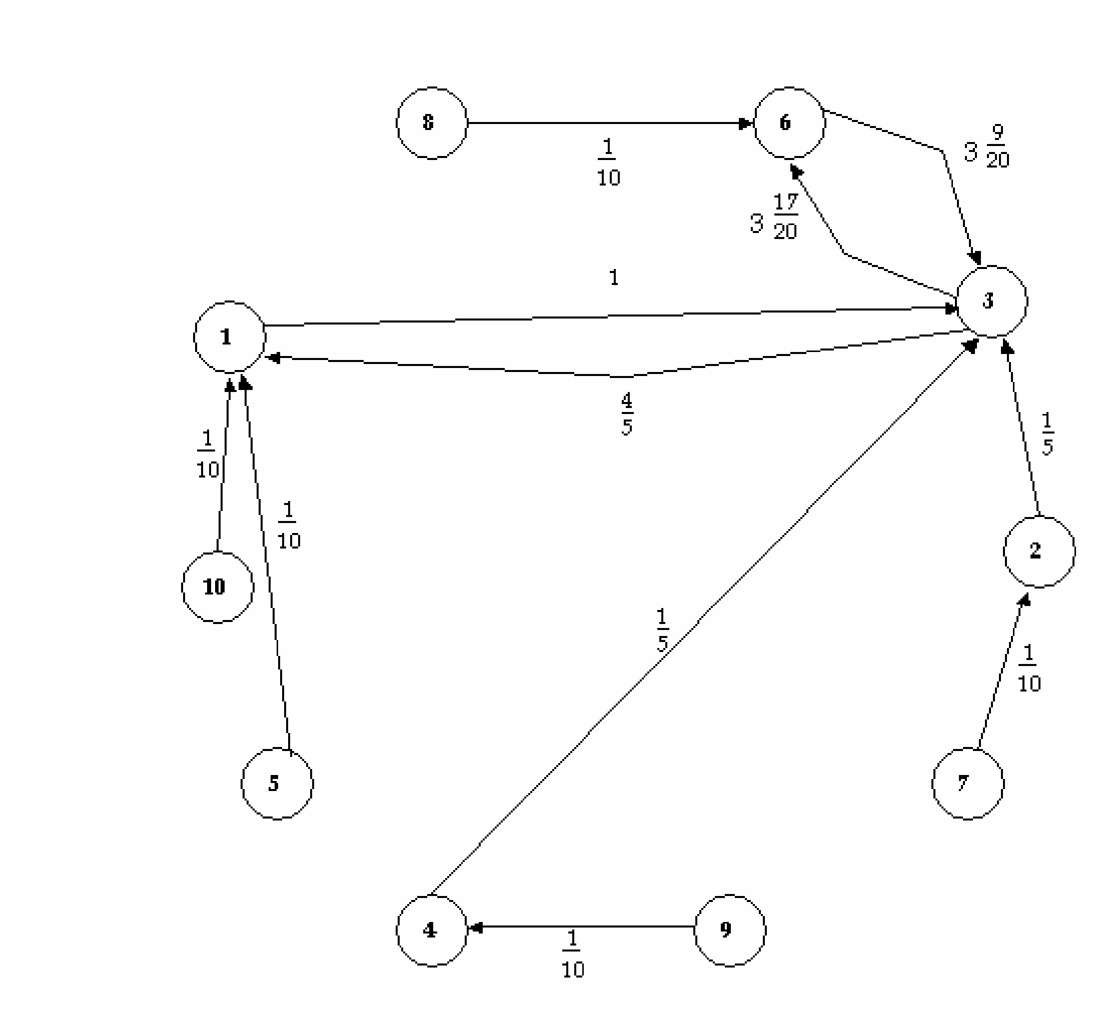
$$\boldsymbol{y}_{i1}^{\boldsymbol{t}} = 0, \ \boldsymbol{t} \neq \boldsymbol{n}$$
$$\boldsymbol{y}_{1\boldsymbol{j}}^{\boldsymbol{t}} = 0, \ \boldsymbol{t} \neq 1$$
$$\boldsymbol{y}_{\boldsymbol{ij}}^{1} = 0, \ \boldsymbol{i} \neq 1$$

$0(\boldsymbol{n})$ Constraints $= \boldsymbol{n}$
$0(\boldsymbol{n}^3)$ Variables $= \boldsymbol{n}^2(\boldsymbol{n}-1)$

Also convenient to introduce $\boldsymbol{x}_{ij}$ variables with constraints

$$\boldsymbol{x}_{\boldsymbol{ij}} = \sum_{\boldsymbol{t}} \boldsymbol{y}_{\boldsymbol{ij}}^{\boldsymbol{t}}$$

FRACTIONAL SOLUTION FROM 1^{ST} (AGGREGATED) TIME-STAGED FORMULATION



Cost = 364.5 (Optimal solution = 881)

NB 'Lengths' of Arcs can be > 1

Second (Fox, Gavish, Graves (1980))

T2: Disaggregate to give

$$\sum_{i \neq j, t} y_{ij}^{t} = 1 \qquad \text{all j}$$

$$\sum_{j \neq i, t} y_{ij}^{t} = 1 \qquad \text{all i}$$

$$\sum_{i, j, i \neq j} y_{ij}^{t} = 1 \qquad \text{all t}$$

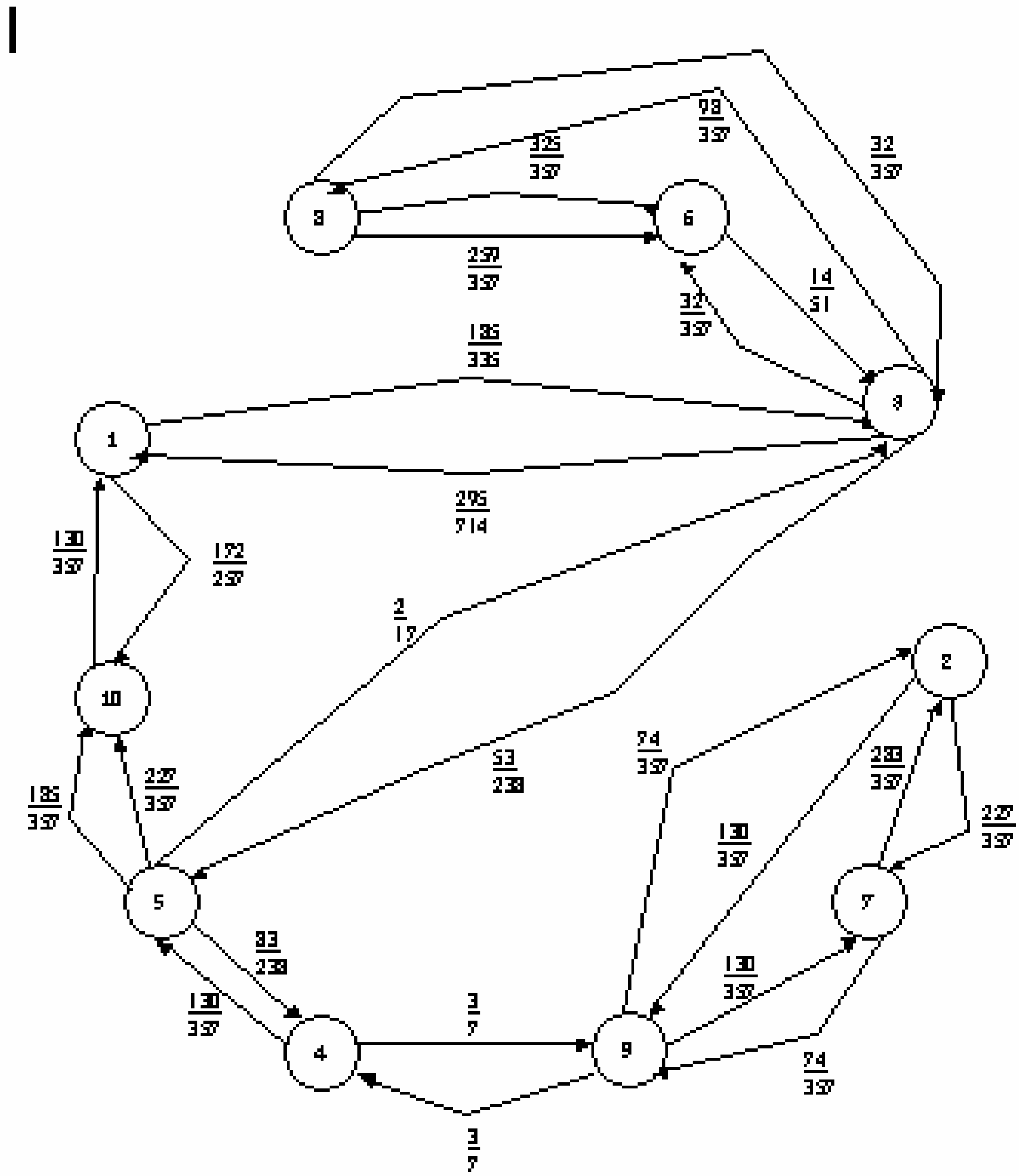
$$\sum_{j=1}^{n} \sum_{t=2}^{n} t y_{ij}^{t} - \sum_{j=1}^{n} \sum_{t=1}^{n} t y_{ji}^{t} = 1 \quad i = 2, 3, \ldots n$$

Initial conditions no longer necessary

0(n) Constraints = *4 n – 1*

0(n³) Variables = *n² (n – 1)*

FRACTIONAL SOLUTION FROM 2nd TIME-STAGED FORMULATION



$$\text{Cost} = 799\frac{164}{357} \quad \text{(optimal solution = 881)}$$

(714 = 2 x 3 x 7 x 17)

Third (Vajda/Hadley (1960))

T3: y_{ij}^{t} interpreted as before

$$\sum_{i \neq j} y_{ij}^{t} = 1 \quad \text{all } j$$

$$\sum_{j \neq i} y_{ij}^{t} = 1 \quad \text{all } i$$

$$\sum_{t} y_{ij}^{t} = 1 \quad \text{all } t$$

$$\sum_{i \neq j} y_{ij}^{t} - \sum_{k \neq j} y_{jk}^{t+1} = 0 \quad \text{all } j, t$$

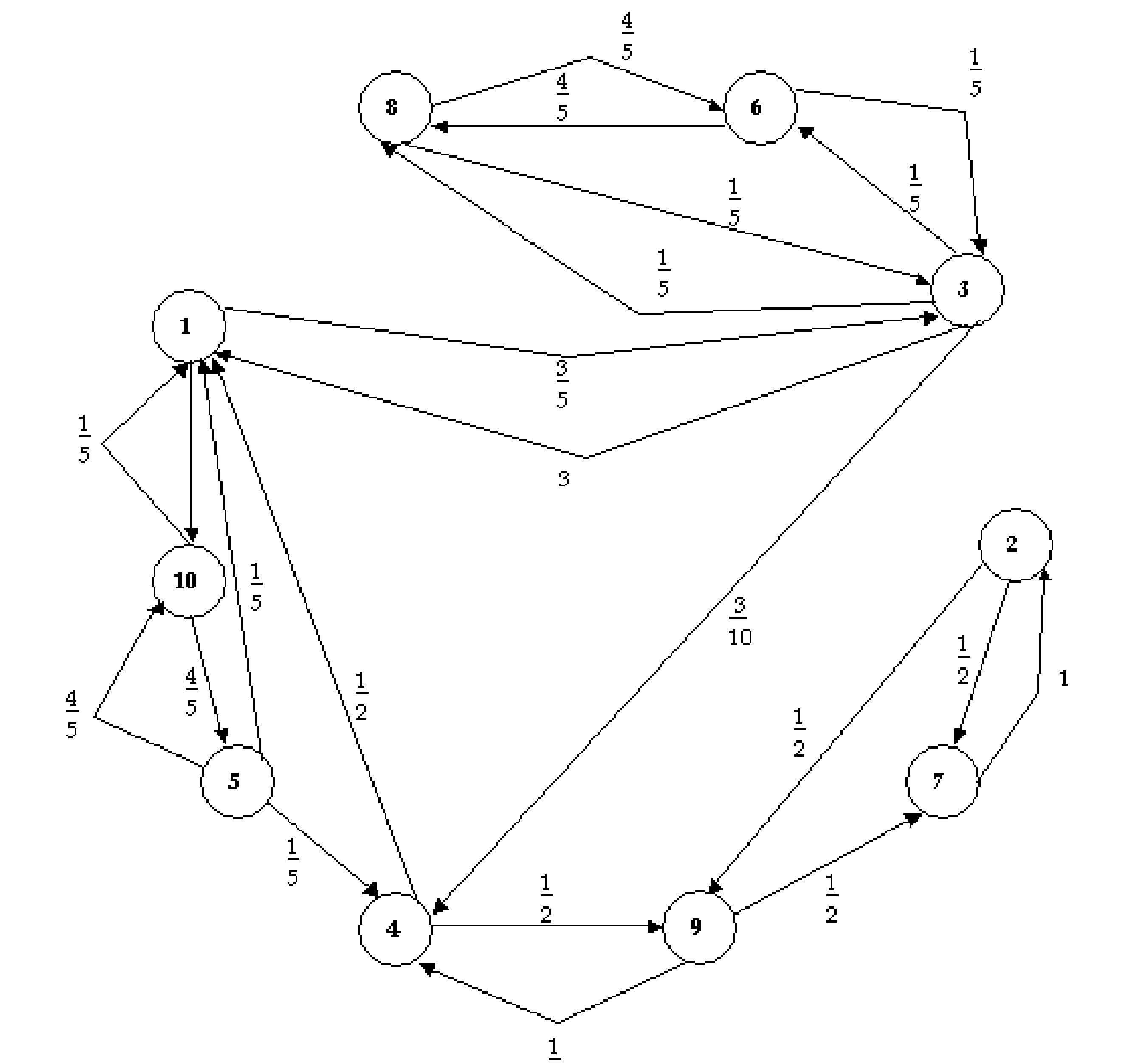
$$\sum_{j \neq 1} y_{1j}^{1} = 1$$

$$\sum_{i \neq 1} y_{i1}^{n} = 1$$

0 (n^2) Constraints $= (2n^2 + 3)$

0 (n^3) Variables $= n^2(n-1)$

FRACTIONAL SOLUTION FROM 2nd TIME 2nd TIME-STAGED FORMULATION



Cost = $804\frac{1}{2}$ Optimal solution = 881

OBSERVATION

Multicommodity Flow Formulation

$$\sum_i y_{ij}^t - \sum_k y_{jk}^t = 0$$

y_{ij}^t is flow $i \to j$ destined for node t

Time Staged Formulation

$$\sum_i y_{ij}^t - \sum_k y_{jk}^{t+1} = 0$$

$y_{ij}^t = 1$ ***iff*** go $i \to j$ at stage t

Are these formulations related?

Can extra variables (y_{ij}^t), introduced *syntactically*, be given different *semantic* interpretations?

COMPARING FORMULATIONS

Minimise: $c\,x$

Subject to: $Ax + By \leq b$

$\underline{x}, \underline{y} \geq 0$, x integer

$$W = \{w \mid wB \geq 0, \underline{w} \geq 0\}$$

W forms a cone which can be characterised by its extreme rays giving matrix Q such that

$$QB \geq 0$$

Hence $QAx \leq Qb$

This is the projection of formulation into space of original variables x_{ij}

COMPARING FORMULATIONS

Project out variables by Fourier-Motzkin elimination to reduce to space of conventional formulation.

P (r) is polytope of LP relaxation of projection of formulation r.

Formulation S (Sequential)

Project out around each *directed cycle* S by summing

$$u_i - u_j + nx_{ij} \leq n - 1$$

$$\downarrow$$

$$n \sum_{i,j \in S} x_{ij} \leq (n-1)|S|$$

ie $\displaystyle\sum_{i,j \in S} x_{ij} \leq |S| - \frac{|S|}{n}$ weaker than |S|-1 (for S a subset of nodes)

Hence $P(S) \supset P(C)$

Formulation F1 (1 Commodity Network Flow)

Projects to $\sum_{ij\in s} x_{ij} \le |S| - \frac{|S|}{n-1}$ stronger than $|S| - \frac{|S|}{n}$

Hence $P(S) \supset P(F1) \supset P(C)$

Formulation F1' (Amended 1 Commodity Network Flow)

Projects to $\frac{1}{n-1} \sum_{\substack{j\in \bar{S}-\{1\} \\ j\in S}} x_{ij} + \sum_{i,j\in S} x_{ij} \le |S| - \frac{|S|}{n-1}$

Hence $P(S) \supset P(F1) \supset F(F1') \supset P(C)$

Formulation F2 (2 Commodity Network Flow)

Projects to $\sum_{i,j} x_{ij} \le |S| - \frac{|S|}{n-1}$

Hence $P(F2) = P(F1)$

<u>Formulation F3</u> (Multi Commodity Network Flow)

Projects to $$\sum_{\substack{i,j \\ \in S}} x_{ij} \leq |S| - 1$$

Hence $P(F3) = P(C)$

<u>Formulation T1</u> (First Stage Dependant)

Projects to

$$\sum_{\substack{i \in S \\ j \in \bar{S} - \{1\}}} x_{ij} \geq \frac{|S|}{n-1}$$

$$\sum_{i,j \in N} x_{ij} = n$$

(Cannot convert 1st constraint to ' $\sum_{i,j \in S} x_{ij} \leq$ 'form since Assignment Constraints not present)

<u>Formulation T2</u> (Second Stage Dependant)

Projects to

$$\frac{1}{n-1}\sum_{\substack{i\in S\\ j\in \bar{S}-\{1\}}} x_{ij} + \frac{1}{n-1}\sum_{\substack{i\in \bar{S}-\{1\}\\ j\in S}} x_{ij} + \sum_{ijj\in s} x_{ij} \leq |S| - \frac{|S|}{n-1}$$

+ others

Hence $P(T2) \subset P(F1')$

<u>Formulation T3</u> (Third Stage Dependant)

Projects to

$$\frac{1}{n-1}\sum_{\substack{i\in S\\ j\in \bar{S}-\{1\}}} x_{ij} + \frac{1}{n-1}\sum_{\substack{i\in \bar{S}-\{1\}\\ j\in S}} x_{ij} + \sum_{n,j\in S} x_{ij} \leq |S| - \frac{|S|}{n-1}$$

+ others

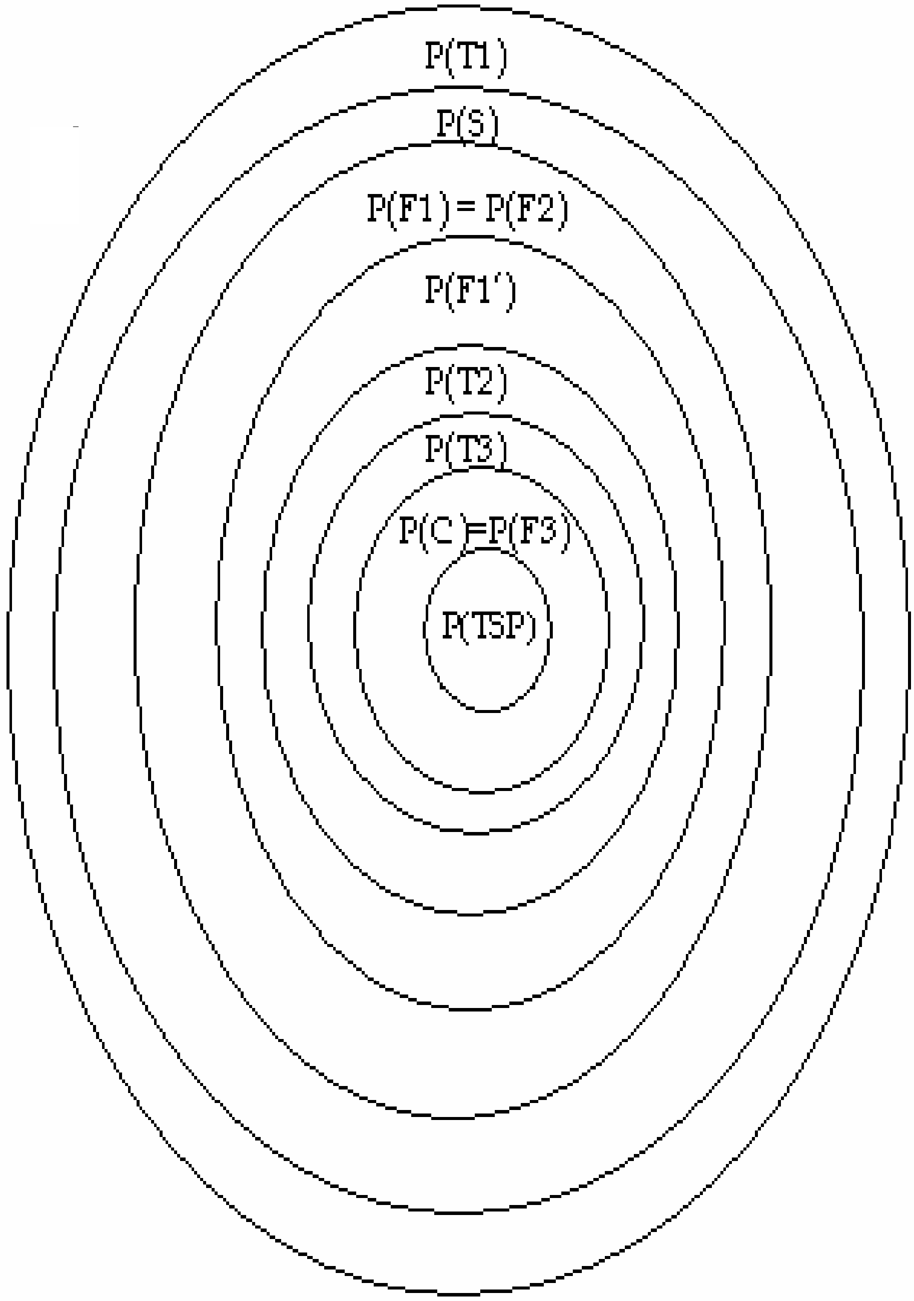
Can show stronger than T2

Hence $P(T3) \subset P(T2)$

Computational Results of a 10-City TSP in order to compare sizes and strengths of LP Relaxations

Model	**Size**	**LP Obj**	**Its**	**Secs**	**IP Obj**	**Nodes**	**Secs**
C (Conventional	502x90						
	(Ass. Relax	766	37	1	766	0	1
	+Subtours (5)	804	40	1	804	0	1
	+Subtours (3)	835	43	1	835	0	1
	+Subtours (2)	878	48	1	881	9	1
S (Sequential)	92x99	773.6	77	3	881	665	16
F1 (Commodity Flow	120x180	794.22	148	1	881	449	13
F' (F1 Modified)	120x180	794.89	142	1	881	369	11
F2 (2 Commodity Flow)	140x270	794.22	229	2	881	373	12
F3 (Multi Commodity Flow)	857x900	878	1024	7	881	9	13
T1 (1st Stage Dependent)	90x990 (10)x(900)	364.5	63	4	881	∞	∞
T2 (2nd Stage Dependent)	120x990 (39) x (900)	799.46	246	18	881	483	36
T3 (3rd Stage Dependent)	193x990 (102)x(900)	804.5	307	5	881	145	27

Solutions obtained using NEW MAGIC and EMSOL



P(TSP) TSP Polytope – not fully known

P(X) Polytope of Projected LP relaxations